Complexity: an approach through independence

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The complexity of an edge of a clutter is the ratio of the size of its minimum subset, that is present only in this given edge, to the size of the edge. The complexity of a clutter is the maximum of the complexities of its edges. We study the complexity of clutters arising from independent sets and matchings of graphs.

Dedicated to Professor Stepan E. Markosyan

The graphs considered in this paper are finite, undirected and do not contain multiple edges or loops. For a graph G let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. For a vertex $v \in V(G)$ let $d(v)$ denote the degree of v, and let $\Delta(G)$ be the maximum degree of a vertex of G. If $E \subseteq E(G)$ then let $V(E)$ be the set of vertices of G which are incident to an edge from E. For $S \subseteq V(G)$ let $G[S]$ denote the subgraph of G induced by the set S.

If u, v are vertices of a graph G, then let $\rho(u, v)$ denote the distance between the two vertices, and let $diam(G)$ denote the diameter of G.

For a positive integer n let K_n denote the complete graph on n vertices. If m and n are positive integers then assume $K_{m,n}$ to be the complete bipartite graph one side of which has m vertices and the other side n vertices.

We also consider clutters. A clutter L, is a pair (V, E), where V is a finite set and E is a family of subsets of V none of which is a subset of another. Following [2], the elements of V will be called vertices of L, while the elements of E-edges of L.

If for a graph $G = (V, E)$ we denote the set of all maximal independent sets of the graph G by U_G, then (V, U_G) is a clutter. In our paper we identify the set U_G and the clutter (V, U_G), and use the same notation U_G for both of them.

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If $L = (V, E)$ is a clutter and $e \in E$ then a set $e_0 \subseteq e$ is called recognizing for e if for any $e' \in E$ with $e_0 \subseteq e'$ we have $e' = e$. In the paper, when we deal with an edge $e \in E$, we denote by S_e one of the smallest recognizing sets for e. Now, for any $e \in E$ set:

$$c(e) = \frac{|S_e|}{|e|},$$

and define $c(L)$, the complexity of the clutter L, as:

$$c(L) = \max_{e \in E} c(e).$$

Note that for every clutter L $0 \leq c(L) \leq 1$.

There are many clutters associated to graphs, nevertheless, in the paper, we define $c(G)$-the complexity of a graph G, as $c(G) = c(U_G)$.

Though the concept of this kind of complexity of a graph is new in graph theory (see [1, 4] for another approach), it can be easily seen that it stems from the well-known "continuation" problems, that is problems, where we are given some initial properties of large objects and asked for the construction of such objects if they exist. A typical example of a problem of this kind may be the finding of a proper edge-coloring of a graph in which a fixed edge (or edges) has a prescribed color.

Though mentioned problems ask for the construction of objects with these properties, in this paper we are interested in "measuring" the ability of an object to be built from its parts, and as a measure we take the ratio of the size of "minimum information" that suffices for identifying the object to the size of the object.

A good way to think about this is to imagine that we are working in some universe, and we need to identify an object from large set of objects that belong to the universe, which in our case is a clutter. Of course, in general, we do not need to have the whole object until we claim that we have identified the object, since it suffice to have only its smallest recognizing subset. The complexity of an object shows the relative information that one needs for the identification, and the complexity of the universe is just the maximum complexity of an object from the universe. In our terminology, the universe which contains an object that requires itself for its identification is considered to be hard or just unrecognizable.

Terms and concepts that we do not define can be found in [2, 5, 7].

1. The complexity in general case

The following proposition can be verified easily:

Proposition 1 *Any independent set of vertices of the graph G can be extended to a member of U_G.*

Proposition 2 *For any $x \in V(G)$ there is $U_x \in U_G$ such that $x \in U_x$.*

Proof. $\{x\}$ is an independent set. Apply proposition 1.

Corollary 1 *For any graph G*

$$\min_{U \in U_G} |U| + \Delta(G) \leq |V(G)|.$$

Proof. Choose a vertex $x \in V(G)$ with $d(x) = \Delta(G)$. Due to proposition 2 there is $U_x \in U_G$ containing x. Note that the neighbours of x do not belong to U_x, thus

$$\min_{U \in U_G} |U| \leq |U_x| \leq |V(G)| - \Delta(G)$$

or,

$$\min_{U \in U_G} |U| + \Delta(G) \leq |V(G)|.$$

Lemma 1 *For any graph G with $|E(G)| \geq 1$ we have:*

$$c(G) \geq \frac{1}{|V(G)| - \Delta(G)}.$$

Proof. Choose $U_0 \in U_G$ with $|U_0| = \min_{U \in U_G} |U|$. Due to corollary 1 $|U_0| \leq |V(G)| - \Delta(G)$. Clearly,

$$c(G) \geq c(U_0) = \frac{|S_{U_0}|}{|U_0|} \geq \frac{1}{|V(G)| - \Delta(G)}.$$

Lemma 2 *A set $U_0 \subseteq U$ is recognizing for $U \in U_G$ if and only if each vertex $v \in V(G) \backslash U$ has a neighbour in U_0.*

Proof. Necessity. If there were a vertex $v \in V(G) \backslash U$ without a neighbour from U_0, then due to proposition 1 we would find $U' \in U_G$ such that $U_0 \cup \{v\} \subseteq U'$. Note that $U' \neq U$ since $v \notin U$. Taking into account that $U_0 \subseteq U$ and $U_0 \subseteq U'$, we deduce that the set U_0 is not recognizing for U.

Sufficiency. Suppose that the set U_0 is not recognizing for U. Then there is $U' \in U_G$, $U' \neq U$ such that $U_0 \subseteq U'$. Since $U' \neq U$ there is $v \in U' \backslash U$. Note that the vertex v has no neighbour from the set U_0. Contradiction.

Corollary 2 *If $U \in U_G$ and there is a vertex $v \in V(G) \backslash U$ that has only one neighbour u in the set U, then all recognizing sets of U contain the vertex u.*

Lemma 3 *Let $G = (V, E)$ be a connected graph such that for each $U \in U_G$ we have $|S_U| = 1$. Then:*

(a) for each $U \in U_G$ and its smallest recognizing set S_U, the vertex from S_U is adjacent to all vertices outside U;

(b) $\min_{U \in U_G} |U| + \Delta(G) = |V(G)|$;

(c) If $U_G = \{U_1, ..., U_l\}$, $S_G = \{v \in V(G) : v$ lies in exactly one $U \in U_G\}$ and for $i = 1, ..., l$ $S_G(U_i) = \{x \in S_G : x \in U_i\}$, then any vertices $u_1, ..., u_l$ with $u_i \in S_G(U_i)$ induce a maximum clique of G, and vice versa, any maximum clique of G can be obtained in this way;

(d) $diam(G) \leq 3$.

Proof. (a) directly follows from lemma 2.

(b)Choose $U_0 \in U_G$ with $|U_0| = \min_{U \in U_G} |U|$. According to (a) there is $x \in U_0$ that is adjacent to all vertices from $V(G) \backslash U$. Note that

$$\Delta(G) \geq d(x) = |V(G)| - |U_0| = |V(G)| - \min_{U \in U_G} |U|$$

thus

$$\Delta(G) \geq |V(G)| - \min_{U \in U_G} |U|.$$

Corollary 1 implies that

$$\Delta(G) + \min_{U \in U_G} |U| = |V(G)|.$$

(c)Let $U_i \in U_G$ and $U_j \in U_G$ $(i \neq j)$, and consider the vertices $v_i \in S_G(U_i)$ and $v_j \in S_G(U_j)$. Clearly, $v_i \notin U_j$ and $v_j \notin U_i$, hence due to (a) $(v_i, v_j) \in E(G)$, thus any vertices $u_1, ..., u_l$ with $u_i \in S_G(U_i), i = 1, ..., l$ induce a clique of G, and particularly, the size of the maximum clique of G is at least l.

This implies that to complete the proof of (c) we only need to show that for any maximum clique Q of the graph G there are $u_1, ..., u_l$ with $u_i \in S_G(U_i), i = 1, ..., l$ such that $V(Q) = \{u_1, ..., u_l\}$.

Let Q be a maximum clique of the graph G, and let $U \in U_G$. Clearly, $|V(Q) \cap U| \leq 1$. Let us show that $|V(Q) \cap U| = 1$. If $V(Q) \cap U = \emptyset$ then due to (a) there is $x \in U$ such that x is adjacent to all vertices of Q, thus the set $V(Q) \cup \{x\}$ forms a larger clique contradicting the choice of Q.

Thus $|V(Q) \cap U| = 1$. Suppose that $V(Q) \cap U = \{x\}$. Let us show that $x \in S_G(U)$. Suppose not. Then there is $U' \in U_G, U' \neq U$ such that $x \in U'$. Clearly, $V(Q) \cap U' = \{x\}$. (a) implies that the vertices $u \in S_U$ and $u' \in S_{U'}$ are adjacent to all vertices lying outside U and U', respectively. Since $x \in U, U'$, we imply that u and u' do not belong to the clique Q. It is also clear that $u \notin U'$ and $u' \notin U$. Now, it is not hard to see that the set $(V(Q) \backslash \{x\}) \cup \{u, u'\}$ induces a clique that is larger than Q contradicting the choice of Q. Thus $x \in S_G(U)$ and the proof of (c) is completed.

(d)Suppose that $diam(G) \geq 4$, and consider the vertices $u, v \in V(G)$ with $\rho(u, v) = diam(G) \geq 4$. Let $u = u_0, u_1, ..., u_k = v$, $k = \rho(u, v) \geq 4$ be the shortest path connecting the vertices u and v. Note that $(u_1, u_3) \notin E(G)$, thus due to proposition 1, there is $U \in U_G$ with $\{u_1, u_3\} \subseteq U$. (a) implies that there is a vertex $z \in U$ such that $(u, z) \in E(G)$ and $(u_4, z) \in E(G)$ which contradicts the choice of the path $u = u_0, u_1, ..., u_k = v$ as the shortest path connecting the vertices u and v. The proof of the lemma 3 is completed.

Theorem 1 *If $G = (V, E)$ is a connected graph with $|V(G)| \geq 2$ that is not isomorphic to $K_{2,2}, K_{3,3}, K_{4,4}$ then*

$$c(G) \geq \frac{1}{1 + |V(G)| - 2\sqrt{|V(G)| - 1}}.$$

Proof. Suppose that there is $U \in U_G$ with $|S_U| \geq 2$. Since G is connected and $|V(G)| \geq 2$, we have $|U| \leq |V(G)| - 1$, thus

$$c(G) \geq c(U_0) = \frac{|S_{U_0}|}{|U_0|} \geq \frac{2}{|V(G)| - 1} \geq \frac{1}{1 + |V(G)| - 2\sqrt{|V(G)| - 1}}.$$

Note that the final inequality is true because it is equivalent to $2+2|V(G)|-4\sqrt{|V(G)|-1} \geq |V(G)| - 1$, or $|V(G)| + 3 \geq 4\sqrt{|V(G)| - 1}$, which in its turn follows from

$$|V(G)|^2 + 6|V(G)| + 9 \geq 16|V(G)| - 16,$$

or

$$|V(G)|^2 - 10|V(G)| + 25 = (|V(G)| - 5)^2 \geq 0.$$

Thus, without loss of generality, we may assume that for each $U \in U_G$ we have $|S_U| = 1$. Note that if we could prove that in such graphs

$$|V(G)| \leq 1 + \left(\frac{1 + \Delta(G)}{2}\right)^2, \tag{1}$$

which is equivalent to

$$\Delta(G) \geq 2\sqrt{|V(G)| - 1} - 1,$$

then, due to lemma 1, we would have

$$c(G) \geq \frac{1}{1 + |V(G)| - 2\sqrt{|V(G)| - 1}},$$

and the proof of the theorem will be completed. Thus, to complete the proof, it suffices to show that if G is a graph satisfying the conditions of the theorem 1 and for each $U \in U_G$ we have $|S_U| = 1$, then the inequality (1) holds.

Let $U_G = \{U_1, ..., U_l\}$, and suppose Q is a maximum clique of G with $V(Q) = \{v_1, ..., v_l\}$, $v_i \in S_G(U_i), i = 1, ..., l$ ((c) of lemma 3). Set: $V_0 = V(G) \backslash V(Q)$.

Let us show that each $x \in V_0$ has a neighbour in Q. Since G is connected and $|V(G)| \geq 2$, there is $y \in V(G)$ such that $(x, y) \in E(G)$. Due to proposition 2, there is $U_y \in U_G$ containing the vertex y. Due to (a) and (c) of lemma 3 there is $z \in V(Q) \cap U_y$ such that z is adjacent to all vertices lying outside U_y, and particularly, to x.

Then, let us show that, without loss of generality, we may assume that $l \geq 3$. Suppose that $l = 2$. (c) of lemma 3 implies that G does not contain a triangle. We claim that G is bipartite. Suppose not, and let C be the shortest odd cycle of the graph G, with $V(C) = \{z_1, ..., z_k\}$, $E(C) = \{(z_1, z_2), ..., (z_{k-1}, z_k), (z_k, z_1)\}$ and $k \geq 5$. Since C is the shortest odd cycle, we have $(z_1, z_4) \notin E(G)$. Due to proposition 1, there is $U_{z_1,z_4} \in U_G$ containing the vertices z_1 and z_4. (a) of lemma 3 implies that there is $x \in U_{z_1,z_4}$ that is adjacent to all vertices lying outside U_{z_1,z_4}. Since $z_2 \notin U_{z_1,z_4}$ and $z_3 \notin U_{z_1,z_4}$, we have $(x, z_2) \in E(G)$ and $(x, z_3) \in E(G)$. This is a contradiction since the vertices x, z_2, z_3 induce a triangle.

Thus G is a bipartite graph, and let (X_1, X_2) be the bipartition of G, where $V(G) = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$. It is clear that $X_1 \in U_G$ and $X_2 \in U_G$. Since, by assumption $l = 2$, we have $U_G = \{X_1, X_2\}$. This, and proposition 1 imply that for each $x_1 \in X_1$ and $x_2 \in X_2$ we have $(x_1, x_2) \in E(G)$. Thus the graph G is isomorphic to the complete bipartite graph $K_{m,n}$ for some m, n with $m \geq n$.

Now if $n = 1$ then $|V(G)| = m + 1$, $\Delta(G) = m$, and therefore

$$|V(G)| = m + 1 \leq 1 + \left(\frac{1+m}{2}\right)^2 = 1 + \left(\frac{1+\Delta(G)}{2}\right)^2,$$

thus, we can assume that $n \geq 2$. On the other hand, if $m \geq 5$ then $|V(G)| = m + n$, $\Delta(G) = m$, and therefore

$$|V(G)| = m + n \leq 2m \leq 1 + \left(\frac{1+m}{2}\right)^2 = 1 + \left(\frac{1+\Delta(G)}{2}\right)^2,$$

thus, we can assume that $m \leq 4$. Since by assumption G is not isomorphic to $K_{2,2}, K_{3,3}, K_{4,4}$ then G is either $K_{2,3}$ or $K_{2,4}$ or $K_{3,4}$. It is a matter of direct verification that these three graphs satisfy the inequality (1).

Thus, we may assume that $l \geq 3$. We will consider four cases.

Case 1: V_0 is an independent set.

Proposition 1 and lemma 3 imply that there is $w \in V(Q)$ such that $(\{w\} \cup V_0) \in U_G$. Note that all neighbours of w belong to Q and $d(w) = l - 1$. Taking into account that all vertices of V_0 are adjacent to a vertex from Q, we deduce

$$|V_0| \leq (l-1)(\Delta(G) - (l-1)),$$

and therefore

$$|V(G)| = |V_0| + l \leq l + (l-1)(\Delta(G) - (l-1)) = 1 + (l-1)((\Delta(G)+1) - (l-1)) \leq 1 + \left(\frac{1+\Delta(G)}{2}\right)^2.$$

Case 2: $G[V_0]$ contains at least two independent edges.

Suppose that $(x_1, y_1) \in E(G)$, $(x_2, y_2) \in E(G)$, $\{x_1, y_1, x_2, y_2\} \subseteq V_0$ and $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$. Choose a vertex $z \in V(Q)$. Proposition 2 and (c) of lemma 3 imply that there is $U_z \in U_G$ with $\{z\} = S_{U_z}$, therefore $(z, x_i) \in E(G)$ or $(z, y_i) \in E(G)$,for $i = 1, 2$. This implies

$$|V_0| \leq l(\Delta(G) - (l-1)) - 2(l-2),$$

and taking into account that $l \geq 3$, we deduce

$$|V(G)| = |V_0| + l \leq l + l(\Delta(G) - (l-1)) - 2l + 4 = 1 + l(\Delta(G) + 1 - l) - l + 3 \leq$$
$$1 + l(\Delta(G) + 1 - l) \leq 1 + \left(\frac{1+\Delta(G)}{2}\right)^2.$$

Case 3: The maximum number of independent edges in $G[V_0]$ is one, and $G[V_0]$ contains a triangle.

Suppose that $u, v, w \in V_0$ and $(u, v), (u, w), (v, w) \in E(G)$. Define $N(u), N(v), N(w)$ as the sets of neighbours of u, v, w in Q, respectively. Since for each $z \in V(Q)$ there is $U_z \in U_G$ with $\{z\} = S_{U_z}$, we imply

$$N(u) \cup N(v) = V(Q), N(u) \cup N(w) = V(Q), N(v) \cup N(w) = V(Q),$$

and therefore

$$|N(u) \cup N(v)| = l, |N(u) \cup N(w)| = l, |N(v) \cup N(w)| = l.$$

Note that

$$\begin{aligned} l &= |N(u) \cup N(v) \cup N(w)| = |N(u)| + |N(v)| + |N(w)| - |N(u) \cap N(v)| - |N(u) \cap N(w)| \\ &-|N(v) \cap N(w)| + |N(u) \cap N(v) \cap N(w)| = |N(u)| + |N(v)| + |N(w)| - |N(u)| - |N(v)| \\ &+|N(u) \cup N(v)| - |N(u)| - |N(w)| + |N(u) \cup N(w)| - |N(v)| - |N(w)| + |N(v) \cup N(w)| \\ &+|N(u) \cap N(v) \cap N(w)| = 3l + |N(u) \cap N(v) \cap N(w)| - |N(u)| - |N(v)| - |N(w)| \end{aligned}$$

hence, taking into account that $l \geq 3$, we deduce

$$|N(u)| + |N(v)| + |N(w)| = 2l + |N(u) \cap N(v) \cap N(w)| > l + 2.$$

Clearly,

$$|V_0| \leq l(\Delta(G) - (l - 1)) - (|N(u)| - 1 + |N(v)| - 1 + |N(w)| - 1)$$

and therefore

$$\begin{aligned} &|V(G)| = |V_0| + l \leq l + l(\Delta(G) + 1 - l) - |N(u)| - |N(v)| - |N(w)| + 3 < \\ &l(\Delta(G) + 1 - l) - 2 + 3 = 1 + l(\Delta(G) + 1 - l) \leq 1 + \left(\frac{1 + \Delta(G)}{2}\right)^2. \end{aligned}$$

Case 4: The maximum number of independent edges in $G[V_0]$ is one, and $G[V_0]$ contains no triangle.

In this case, there is $x \in V_0$ that is incident to all edges of $G[V_0]$. Suppose that $V_0 = \{x, y_1, ..., y_r, z_1, ..., z_q\}$, $r \geq 1, q \geq 0$, and the vertex x is adjacent to all y_i's and is not adjacent to either of z_j's.

Define the sets $N(x), N(y_1), ..., N(y_r)$ as the neighbours of the vertices $x, y_1, ..., y_r$ in the clique Q, respectively. Since for each $z \in V(Q)$ there is $U_z \in U_G$ with $\{z\} = S_{U_z}$, we imply

$$N(x) \cup N(y_i) = V(Q), \text{for } i = 1, ..., r.$$

Moreover, since the clique Q is maximum, we have:

$$\emptyset \subset N(x) \subset V(Q), \text{and } \emptyset \subset N(y_i) \subset V(Q), \text{for } i = 1, ..., r.$$

If there is i such that $N(x) \cap N(y_i) \neq \emptyset$, then

$$\begin{aligned} &|N(x)| + |N(y_1)| + ... + |N(y_r)| = |N(x) \cup N(y_i)| + |N(x) \cap N(y_i)| + |N(y_1)| + ... + |N(y_{i-1})| \\ &+|N(y_{i+1})| + ... + |N(y_r)| \geq l + 1 + r - 1 = l + r. \end{aligned}$$

Taking into account that

$$|V_0| \leq l(\Delta(G) - (l-1)) - (|N(x)| - 1 + |N(y_1)| - 1 + ... + |N(y_r)| - 1)$$

we get

$$|V(G)| = |V_0| + l \leq l + l(\Delta(G) + 1 - l) - |N(x)| - |N(y_1)| - ... - |N(y_r)| + r + 1 \leq$$
$$1 + l(\Delta(G) + 1 - l) \leq 1 + \left(\frac{1 + \Delta(G)}{2}\right)^2.$$

Thus, without loss of generality, we may assume that for each $i, 1 \leq i \leq r$ we have $N(x) \cap N(y_i) = \emptyset$. Consider an arbitrary edge (x, y_i), $1 \leq i \leq r$. Since for each $z \in V(Q)$ $(z, x) \in E(G)$ or $(z, y_i) \in E(G)$ ((c) of lemma 3), we deduce

$$|V_0| \leq l(\Delta(G) + 1 - l) - (l - 2).$$

Now, if $|V_0| \neq l(\Delta(G) + 1 - l) - (l - 2)$ then

$$|V_0| \leq l(\Delta(G) + 1 - l) - (l - 1)$$

and therefore

$$|V(G)| = |V_0| + l \leq l + l(\Delta(G) + 1 - l) - (l - 1) = 1 + l(\Delta(G) + 1 - l) \leq 1 + \left(\frac{1 + \Delta(G)}{2}\right)^2.$$

Thus, we may also assume that $|V_0| = l(\Delta(G) + 1 - l) - (l - 2)$, which immidiately implies that for each $z \in V(Q)$ we have $d(z) = \Delta(G)$.

Now, note that the sets $\{x, z_1, ..., z_q\}$ and $\{y_1, ..., y_r, z_1, ..., z_q\}$ are independent, hence due to proposition 1 and (a) of lemma 3 there are $w_1 \in V(Q)$ and $w_2 \in V(Q)$ such that $(\{x, z_1, ..., z_q\} \cup \{w_1\}) \in U_G$ and $(\{y_1, ..., y_r, z_1, ..., z_q\} \cup \{w_2\}) \in U_G$. Clearly, $w_1 \neq w_2$. (a) of lemma 3 implies that

$$\Delta(G) = d(w_1) = l - 1 + r, \Delta(G) = d(w_2) = l - 1 + 1,$$

hence $\Delta(G) = l$, and $r = 1$. Taking into account that for each $w \in V(Q)$ $(w, x) \in E(G)$ or $(w, y_1) \in E(G)$ ((c) of lemma 3), we deduce $q = 0$. Hence

$$|V(G)| = l + 2.$$

Taking into account that $\Delta(G) = l \geq 3$ we deduce

$$|V(G)| = l + 2 = 1 + (l + 1) \leq 1 + \left(\frac{1 + l}{2}\right)^2 = 1 + \left(\frac{1 + \Delta(G)}{2}\right)^2.$$

The proof of the theorem 1 is completed.

Remark 1 *There is an infinite sequence of graphs attaining the bound of the theorem 1. For a positive integer n consider the graph G from figure 1. Note that $|V(G)| = 1 + n^2, \Delta(G) = 2n - 1$ and $c(G) = \frac{1}{n^2 - 2n + 2}$.*

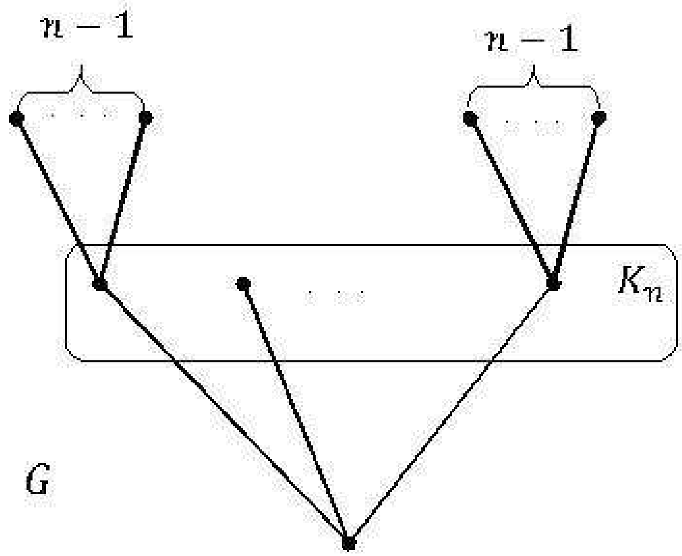

Figure 1. Example attaining the bound of theorem 1

Theorem 2 *For any $m, n \in N$ with $1 \leq m \leq n$ there is a connected bipartite graph G such that $c(G) = \frac{m}{n}$.*

Proof. For any $m, n \in N$ with $1 \leq m \leq n$ consider the connected bipartite graph G from the figure 2.

Define:

$$S = \{s_1, ..., s_{n-m+1}\}, T = \{t_1, ..., t_{n-m+1}\}, X = \{x_1, ..., x_{m-1}\}, Y = \{y_1, ..., y_{m-1}\}.$$

Let us show that $c(G) = \frac{m}{n}$. Choose any $U \in U_G$. We will consider two cases:

Case 1: $a \in U$.

Clearly, for each $s \in S, s \notin U$ and for each $x \in X, x \notin U$, therefore $U = \{a\} \cup T \cup Y$. Lemma 2 implies that $S_U = \{a\}$, thus

$$c(U) = \frac{|S_U|}{|U|} = \frac{1}{n+1}.$$

Case 2: $a \notin U$.

It is clear that

$$|\{x_i, y_i\} \cap U| = 1, \text{ for } i = 1, ..., m-1; \tag{2}$$

$$T \cap U = \emptyset \Leftrightarrow S \cap U = S \Leftrightarrow S \subseteq U; \tag{3}$$

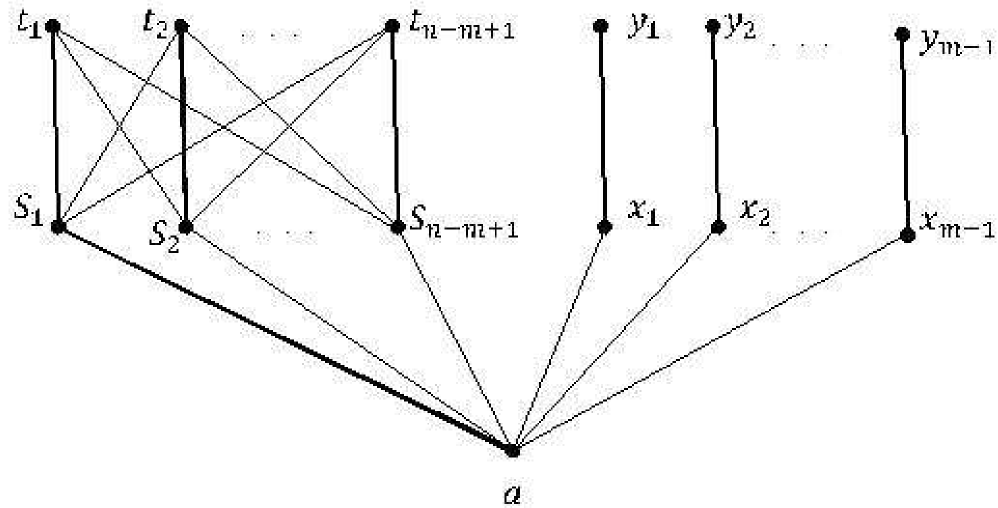

Figure 2. A graph G with $c(G) = \frac{m}{n}$

$$S \cap U = \emptyset \Leftrightarrow T \cap U = T \Leftrightarrow T \subseteq U; \tag{4}$$

(2)-(4) imply that $|U| = n$.

Now, note that if there is $x_i \in U$, then x_i with respect to y_i and U satisfies the conditions of the corollary 2, thus $x_i \in S_U$. Similarly, if there is $y_i \in U$ then y_i with respect to x_i and U satisfies the conditions of the corollary 2 ($a \notin U$), thus $y_i \in S_U$.

On the other hand, if $S \subset U$ then due to (3) $T \cap U = \emptyset$, hence lemma 2 implies that there is $s \in S$ such that $s \in S_U$. Similarly, if $T \subset U$ then there is $t \in T$ such that $t \in S_U$. This implies that there is $s \in S$ such that $(X \cap U) \cup (Y \cap U) \cup \{s\} \subseteq S_U$ or $(X \cap U) \cup (Y \cap U) \cup \{s\} \subseteq S_U$. Now, it is not hard to see that either $(X \cap U) \cup (Y \cap U) \cup \{s\}$ or $(X \cap U) \cup (Y \cap U) \cup \{s\}$ is recognizing for U, hence $(X \cap U) \cup (Y \cap U) \cup \{s\} = S_U$ or $(X \cap U) \cup (Y \cap U) \cup \{s\} = S_U$, and therefore

$$|S_U| = |X \cap U| + |Y \cap U| + 1 = m,$$

since, due to (2), we have $|X \cap U| + |Y \cap U| = m - 1$. Thus:

$$c(U) = \frac{|S_U|}{|U|} = \frac{m}{n}.$$

The considered two cases imply

$$c(G) = max\{\frac{1}{n+1}, \frac{m}{n}\} = \frac{m}{n}.$$

The proof of the theorem 2 is completed.

2. Hardness results for complexity

The aim of this section is the investigation of some problems that are related to the algorithmic computation of the complexity of graphs.

We start with a problem that is related to finding a recognizing set for a given maximal independent set.

Problem 1:
Condition: Given a graph G, $U \in U_G$ and a positive integer k.
Question: Is there a recognizing set $U' \subseteq U$ for U with $|U'| = k$?

Theorem 3 *The* ***Problem 1*** *is NP-complete already for bipartite graphs.*

Proof. Lemma 2 implies that the **Problem 1** belongs to the class NP. To show the completeness of the problem, we will reduce the classical **Set Cover** problem to our problem restricted to bipartite graphs. Recall that the **Set Cover** is formulated as follows ([3]):

Problem: Set Cover
Condition: Given a set $A = \{a_1, ..., a_n\}$, a family $\mathcal{A} = \{A_1, ..., A_m\}$ of subsets of the set A with $A_1 \cup ... \cup A_m = A$, and a positive integer $l, l \leq m$.
Question: Are there $A_{i_1}, ..., A_{i_l} \in \mathcal{A}$ with $A_{i_1} \cup ... \cup A_{i_l} = A$?

For an instance I of **Set Cover** consider the graph $G_I = (V, E)$, where

$$V = \{a_1, ..., a_n, A_1, ..., A_m\}, E = \{(a_i, A_j) : a_i \in A_j, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Note that G_I is bipartite. Consider the set $U = \{A_1, ..., A_m\}$. Since $A_1 \cup ... \cup A_m = A$, we have $U \in U_{G_I}$.

It can be easily verified that the set U has a recognizing subset comprised of l elements if and only if there are $A_{i_1}, ..., A_{i_l} \in \mathcal{A}$ with $A_{i_1} \cup ... \cup A_{i_l} = A$. The proof of the theorem is completed.

Now, we are turning to the investigation of the computation of $c(G)$-the complexity of a graph G. Consider the following

Problem 2:
Condition: Given a graph G and positive integers k, m with $1 \leq k \leq m$.
Question: Does the inequality $c(G) \leq \frac{k}{m}$ hold?

Theorem 4 *The* ***Problem 2*** *is NP-hard already for bipartite graphs.*

Proof. Consider the problem

Problem: Minimum Set Cover
Condition: Given a set $A = \{a_1, ..., a_n\}$, a family $\mathcal{A} = \{A_1, ..., A_m\}$ of subsets of the set A with $A_1 \cup ... \cup A_m = A$ and a positive integer $l, l \leq m$.

Question: Does the size of minimum cover of A, that is, the minimum number l_{min} for which there are $A_{i_1}, ..., A_{i_{l_{min}}} \in \mathcal{A}$ with $A_{i_1} \cup ... \cup A_{i_{l_{min}}} = A$ satisfy the condition $l_{min} \leq l$?

The NP-completeness of **Set Cover** implies that **Minimum Set Cover** is NP-hard, thus to complete the proof of the theorem, it suffices to reduce the **Minimum Set Cover** to **Problem 2** restricted to bipartite graphs.

Given an instance I of **Minimum Set Cover** consider the graph $G_I = (V, E)$, where

$$V = \{A_1, ..., A_m\} \cup \{a_i^{(k)} : 1 \leq i \leq n, 1 \leq k \leq (n+m)^2\},$$
$$E = \{(a_i^{(k)}, A_j) : a_i \in A_j, 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq (n+m)^2\}.$$

Note that G_I is bipartite. Let us show that

$$c(G_I) = \frac{l_{min}}{m},$$

where l_{min} denotes the size of minimum cover of A.

Choose any $U \in U_{G_I}$. We will consider two cases:
Case 1: $U = \{A_1, ..., A_m\}$.
Lemma 2 and the definition of G_I imply that $|S_U| = l_{min}$, therefore

$$c(U) = \frac{l_{min}}{m}.$$

Case 2: $U \neq \{A_1, ..., A_m\}$.
Suppose that $U \cap \{A_1, ..., A_m\} = \{A_{i_1}, ..., A_{i_r}\}$. Since $U \neq \{A_1, ..., A_m\}$, we imply that $A_{i_1} \cup ... \cup A_{i_r} \neq A$. Assume that there are $r', r' \geq 1$ elements of A that do not belong to either of A_{i_j}'s. Note that all $r'(n+m)^2$ copies of these r' elements belong to U, and

$$|U| = r + r'(n+m)^2.$$

On the other hand, if we consider the set $U' \subseteq U$, where

$$U' = \{A_{i_1}, ..., A_{i_r}\} \cup \{a_i^{(1)} : a_i \text{ does not belong to either of } A_{i_j}\text{'s}\},$$

then, according to lemma 2 this would be a recognizing set for U, therefore

$$c(U) = \frac{|S_U|}{|U|} \leq \frac{|U'|}{|U|} = \frac{r + r'}{r'(n+m)^2} \leq \frac{n+m}{(n+m)^2} = \frac{1}{n+m} < \frac{1}{m} \leq \frac{l_{min}}{m}.$$

The considered two cases imply $c(G_I) = \frac{l_{min}}{m}$.

Now, it is not hard to verify that in the instance I of **Minimum Set Cover** $l_{min} \leq l$, if and only if $c(G_I) \leq \frac{l}{m}$. The proof of the theorem is completed.

3. The complexity of trees

In this section we study the complexity of trees.

Definition 1 *In a tree T a vertex $t \in V(T)$ is*

(a) α-vertex, if there is $t' \in V(T)$ with $d(t') = 1$ and $\rho(t, t') = 2$;

(b) β-vertex, if it is adjacent to an α-vertex, whose all neighbours that differ from t, are α-vertices;

(c) γ-vertex, if it is adjacent to a β-vertex;

(d) β-vertex, if it is adjacent to an α-vertex, whose all neighbours that differ from t, are α or γ-vertices;

(e) δ-vertex, if all its neighbours are α or γ-vertices;

Remark 2 *By definition, a vertex of a tree can satisfy more than one of conditions of definition 1, and thus be of more than one type.*

Remark 3 *The definition has a recursive structure, and in (c)- the definition of a γ-vertex - a β-vertex is understood as one which is defined by (b) or (d). For the sake of clear explanation and proving the next lemma, we will imagine that the definition is just a labeling algorithm, which, for its input gets a tree, during the initialization labels all α-vertices according to (a) of definition 1, then at the first step labels all β-vertices and their neighbour γ-vertices according to (b) and (c) of definition 1, respectively. Next, if at the k^{th} step, the labeling is done, then in $(k+1)^{th}$ step it labels all β-vertices and their neighbour γ-vertices according to (d) and (c) of definition 1, respectively. The process continues until no new vertex receives a label. Finally, in the last step, the algorithm labels all δ-vertices according to (e) of definition 1 and presents the labeling of the input tree as the output.*

Remark 4 *By definition, every β-vertex of a tree is a δ-vertex, therefore it is natural to introduce the following*

Definition 2 *A δ-vertex is called pure, if it is not a β-vertex.*

The following lemma explains the essence of definition 1.

Lemma 4 *Let T be a tree, and suppose that $U \in U_T$ and $c(U) = 1$. Then:*

(1) all α-vertices do not belong to U;

(2) all β-vertices belong to U;

(3) all γ-vertices do not belong to U;

(4) all δ-vertices belong to U;

Proof. (1)Suppose that t is an α-vertex. Then, due to (a) of definition 1, there is $t' \in V(T)$ with $d(t') = 1$ and $\rho(t, t') = 2$. If $t \in U$, then the only neighbour of t', which is also a neighbour of t, does not lie in U, hence $t' \in U$ as $U \in U_T$. Now, observe that $U \backslash \{t'\}$ is a recognizing set for U, since it trivially satisfies the condition of the lemma 2. This implies that

$$c(U) = \frac{|S_U|}{|U|} < 1,$$

which is a contradiction.

(2),(3) We will give a simultaneous proof of (2) and (3) by induction on k, where k is the current step of the labeling algorithm (remark 3).

So, assume that $k = 1$, t is a β-vertex and it "became" such a one due to (b) of definition 1. Let us show that $t \in U$.

According to (b) of definition 1, there is an α-vertex t', whose all neighbours except t, are α-vertices. Due to (1) of lemma 4, neither t' nor its α-neighbours that differ from t, do not belong to U. Since $U \in U_T$, we deduce $t \in U$.

This immidiately implies that all γ-vertices that are adjacent to a β-vertex that was labeled in the first step, do not belong to U.

Now, assume that (2) and (3) are true for vertices which receive their labels in the steps up to k. Consider a β-vertex t which gets its label according to (d) of definition 1 in the $(k + 1)^{th}$ step of the labeling algorithm. Let us show that $t \in U$.

According to (d) of definition 1, there is an α-vertex t', whose all neighbours except t, are α or γ-vertices, which have received their labels earlier than the $(k + 1)^{th}$ step. Due to the induction hypothesis and (1) of lemma 4, neither t' nor its α or γ-neighbours that differ from t, belong to U. Since $U \in U_T$, we deduce $t \in U$.

This implies that all γ-vertices that are adjacent to a β-vertex that was labeled in the $(k + 1)^{th}$ step, do not belong to U.

(4)If t is a δ-vertex, then due to (e) of definition 1, and (1) and (3) of lemma 4, all the neighbours of t do not belong to U, hence $t \in U$ as $U \in U_T$.

The proof of the lemma 4 is completed.

The proved lemma immidiately implies the following necessary condition for a tree T to have complexity one.

Corollary 3 *If T is a tree with $c(T) = 1$, then:*

(a) there is no α or γ-vertex, which is also a β or a δ-vertex;

(b) each δ-vertex t is adjacent to an α or a γ-vertex, whose all neighbours except t are neither a β nor a δ-vertex.

Proof. (a) is clear.

(b)On the opposite assumption, consider a δ-vertex t, whose all neighbours are α or γ-vertices ((e) of definition 1), and whose every neighbour is adjacent to a β or a δ-vertex that is different from t. Due to lemma 4, the vertex t and these β or δ-vertices lying on a distance two from t belong to any $U \in U_T$ with $c(U) = 1$. Now, note that $U \backslash \{t\}$ is a

recognizing set for U, since it trivially satisfies the condition of the lemma 2. This implies that

$$c(U) = \frac{|S_U|}{|U|} < 1,$$

which is a contradiction.

Theorem 5 *If a tree T contains neither a β nor a pure δ-vertex, then for each $u \in V(T)$ with $d(u) = 1$ there is $U \in U_T$ with $c(U) = 1$ and $u \in U$.*

Proof. Unfortunately, the proof of existence of such $U \in U_T$ is not easy. This is the main reason that we will give an algorithmic construction of such $U \in U_T$.

Given $u \in V(T)$ with $d(u) = 1$, we will assume that T is represented as a tree rooted at u.

Step 0:

$$U := \{u\}, Spec := \{\text{the neighbours of } u\}$$

Consider the sets $B_1, ..., B_k$ of vertices lying at a distance three from u, where it is assumed that the vertices of $B_j, 1 \leq j \leq k$ are adjacent to the same vertex. Let $List$ be a list comprised of the sets $B_1, ..., B_k$. Note that since T does not contain a β-vertex, we have that all of $B_1, ..., B_k$ contain a non-α vertex.

Step 1: while $List \neq \emptyset$
remove the first element B of $List$.
Define $A = \{v \in B : v \text{ is not a } \alpha\text{-vertex}\}$
$A' = \{v \in A : \text{ all children of } v \text{ are } \alpha\text{-vertices}\}$

Case 1: $A' \neq \emptyset$
$U := U \cup A'$
Add all children of vertices from A' (which are α -vertices, by definition) to the set $Spec$.

Note that, by definition of A', for each $w \in A \backslash A'$ the set B_w, which is the set of children of w, contains a non-α vertex. Moreover, for each $z \in A'$ if we consider the sets $B_{z_1}, ..., B_{z_s}$ of vertices lying at a distance three from z (the vertices of $B_{z_j}, 1 \leq j \leq s$ are adjacent to the same vertex), then since T contains no δ-vertex, each of these sets contains a non-α vertex.

Add all $B_w, B_{z_1}, ..., B_{z_s}$ to $List$;

Case 2: $A' = \emptyset$
Take any $w \in A$.
$U := U \cup \{w\}$; add the parent x of w to the set $Spec$.

Note that $A' = \emptyset$ implies that for each $y \in B\backslash\{w\}$ the set B_w of children of y contains a non-α vertex. On the other hand, since T contains no β-vertex, then for each $z \in B\backslash A$ the set B_z of children of z contains a non-α vertex.

Add all B_w, B_z to *List*;

Consider the sets B_i of vertices lying at a distance three from w, where it is assumed that B_i is the set of children of z_i.

Case 2.1: B_i contains a non-α vertex;

Add B_i to *List*;

Case 2.2: All vertices of B_i are α-vertices;

$U := U \cup \{z_i\}$; $Spec := Spec \cup B_i$;

Consider the sets $B_{z_1}^{(i)}, ..., B_{z_s}^{(i)}$ of vertices lying at a distance three from z, where we assume that $B_{z_j}^{(i)}$ coincides with the set of children of a vertex ${z_j}^{(i)}$. Since T contains no δ-vertex, then each $B_{z_j}^{(i)}$ contains a non-α vertex.

Add $B_{z_1}^{(i)}, ..., B_{z_s}^{(i)}$ to *List*;

The description of the algorithm is completed.

Let us note that if the algorithm cannot choose the set A then the last vertex from which it is impossible to choose a vertex lying on a distance three, is either a pendant vertex, which has a specific vertex in the set *Spec*, or is a vertex that is adjacent to a pendant vertex, and this pendant vertex will be the specific vertex for it.

It can be easily seen that the algorithm constructs a maximal independent set U of T containing the vertex u. The construction of the set *Spec* implies that each vertex $v \in U$ has a specific neighbour in *Spec*, that is, a neighbour, which is not adjacent to any other vertex of U. This and corollary 2 imply that the complexity of U is one. The proof of the theorem 5 is completed.

Remark 5 *The theorem 5 presents merely a sufficient condition. The trees from figure 3 contain a pure δ-vertex, do not contain a β vertex, and nevertheless, the first of them has a complexity that is less than one, while the second one is of complexity one. On the other hand, the graphs from figure 4 contain a β-vertex, do not contain a pure δ vertex, and nevertheless, the first of them has a complexity that is less than one, while the second one is of complexity one.*

4. The case of line graphs

Below we investigate the complexity in the class of line graphs. This class is interesting not only for its own sake, but also for its connection with another clutter related to graphs. Taking into account, that the clutter U_G of a line graph G coincides with the clutter of maximal matchings of some graph, in this sections we will directly work with the latter clutter without remembering that it was originated from a line graph.

In this section the word "complexity" should be understood as the complexity of the clutter of maximal matchings of a graph.

4.1. Structural Lemmata

Lemma 5 *For every maximal matching H*

1. *The vertices of $V(H \backslash S_H)$ can only be connected to the vertices of $V(S_H)$.*

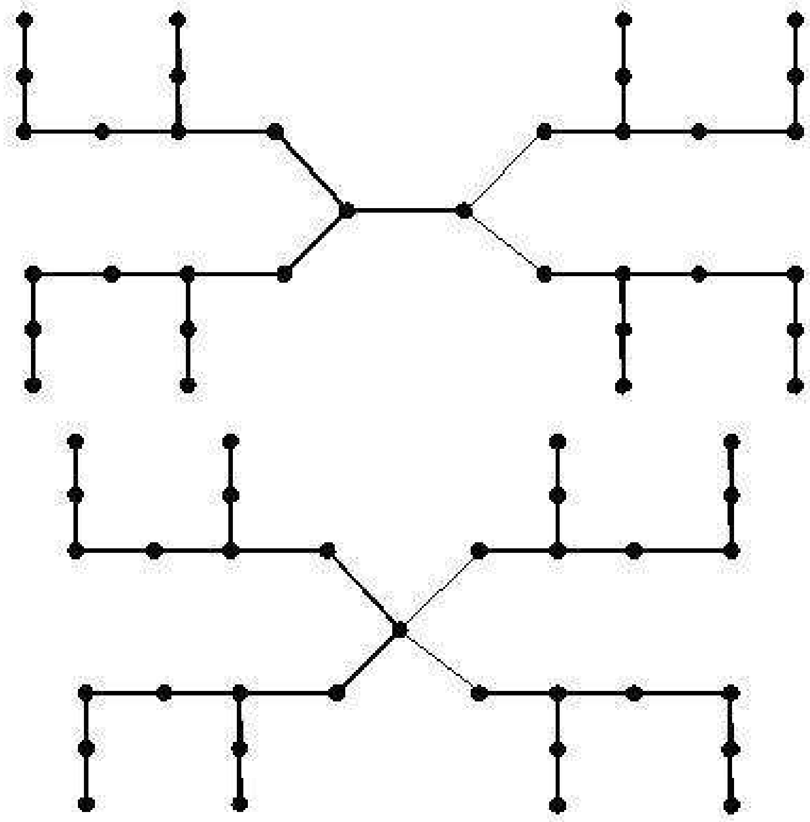

Figure 3. Trees with pure δ-vertices, without β-vertices

2. Each edge in S_H has at least one endpoint connected to a vertex not in $V(S_H)$.

Proof. Let $e = (u, v)$ be an edge in $H \backslash S_H$. Let us first prove that both u and v are not connected to vertices, which are not covered by H. If this is not true, then without loss of generality we may assume that $\exists p \in V(G) \backslash V(H)$, such that $(p, u) \in E(G)$. $H \cup \{(p, u)\} \backslash \{(u, v)\}$ is a maximal matching containing S_H. This contradicts the definition of S_H.

We have proven that the vertices of $V(H \backslash S_H)$ can only be connected to the vertices of $V(H)$.

Now, if there are there are vertices $\{u_1, u_2, u_3, u_4\}$, such that

$$(u_1, u_2), (u_3, u_4) \in H \backslash S_H$$

and $(u_2, u_3) \in E(G)$, then there is a maximal matching that contains $H \backslash \{(u_1, u_2), (u_3, u_4)\} \cup \{(u_2, u_3)\}$. That maximal matching is different from H and contains S_H. This is a contradiction proving point 1.

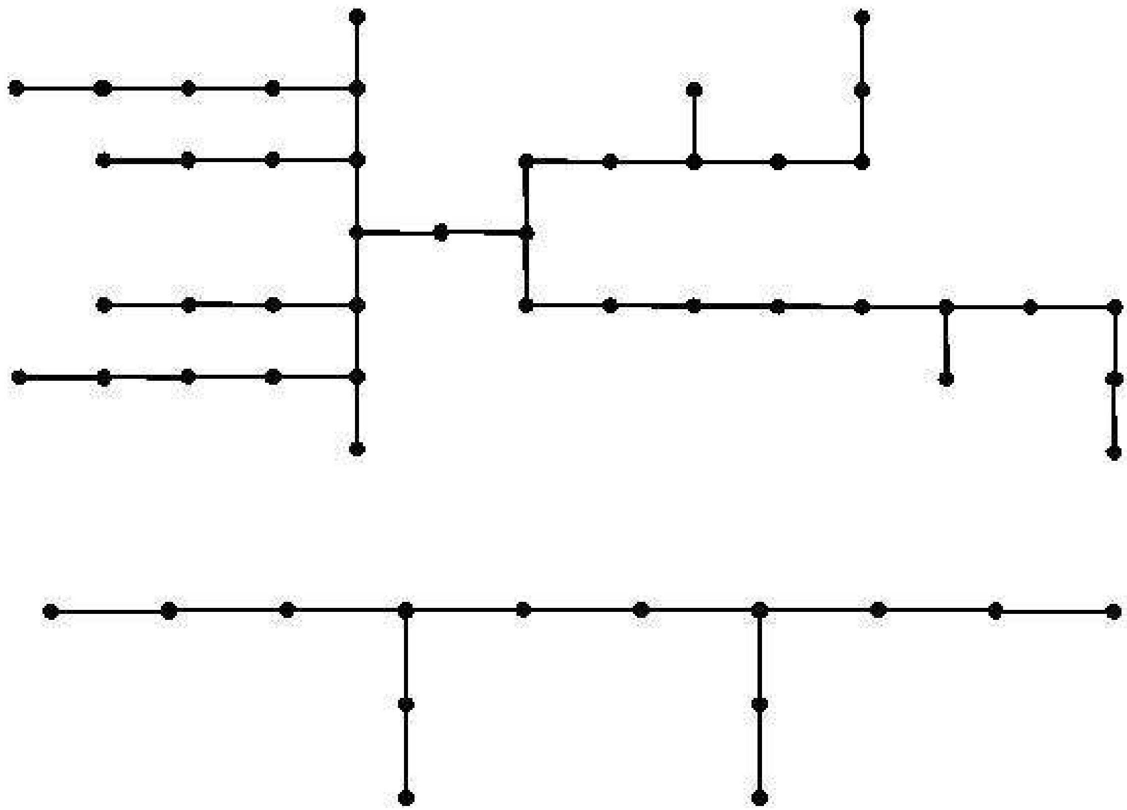

Figure 4. Trees with β-vertices, without pure δ-vertices

If the statement of point 2 does not take place for an edge e, then every maximal matching, which contains $S_H \backslash \{e\}$ also contains S_H. Thus H is the only maximal matching, which contains $S_H \backslash \{e\}$, and consequently S_H is not a minimum subset of H with this property. The contradiction proves point 2.

Lemma 6 *Suppose H is a minimum maximal matching in G and $e \in H$. The endpoints of e cannot be connected to endpoints of different edges of $H \backslash S_H$.*

Proof. Let (u, v) be an edge in S_H. If there are edges (u_1, v_1) and (u_2, v_2) from $H \backslash S_H$, such that u is connected to u_1 and v is connected to u_2, then H is not a minimum maximal matching since the cardinality of

$$H \cup \{(u, u_1), (v, u_2)\} \backslash \{(u, v), (u_1, v_1), (u_2, v_2)\}$$

is less than that of H.

Also, recall the following result [5, 6]:

Lemma 7 *If G is a connected graph, whose every maximal matching is a perfect matching, then G is either K_{2n} or $K_{n,n}$.*

4.2. A lower bound for complexity

Note that the complexity of disconnected graphs does not have a lower bound better than zero. For instance, the graph that consists of a single matching, has a complexity of 0. Moreover, it can be shown that for every rational number r, $0 \le r \le 1$ there exists a graph with complexity r. To construct one just consider the graph G_r from figure 5, where we assume that $r = \frac{a+1}{b+1}$.

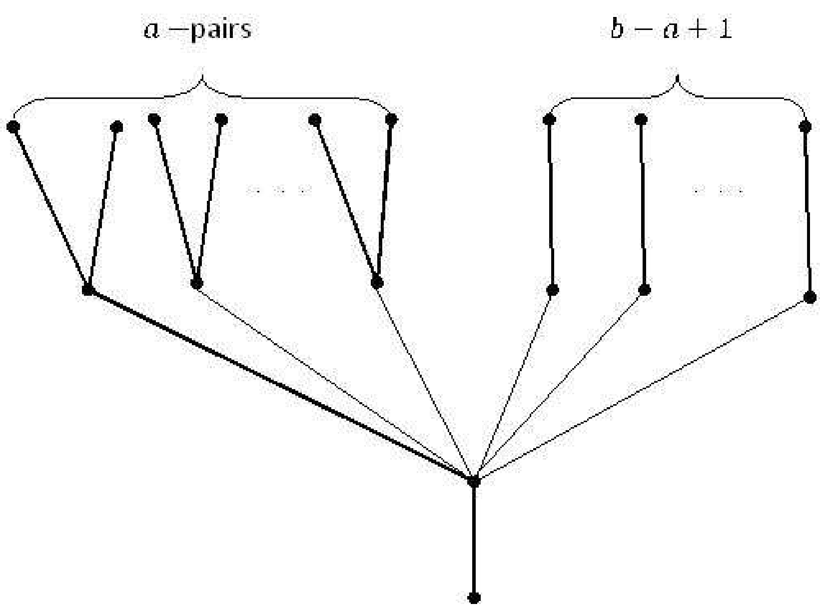

Figure 5. A graph of complexity r

The following theorem proves the existence of a lower bound for the complexities of connected graphs. Before we move on, let us note that the bound given in the theorem below, is significantly better than the one that theorem 1 provides.

Theorem 6 *For every connected graph G with $|V(G)| > 4$ $c(G) \geq \frac{2}{|V(G)|-2}$.*

Proof. Let H be a minimum maximal matching of G. If $|H| < \lfloor |V|/2 \rfloor$ then $|H| \le \frac{|V|-2}{2}$ and

$$c(G) \geq c(H) = \frac{|S_H|}{|H|} \geq \frac{1}{(|V|-2)/2} = \frac{2}{|V|-2} \tag{5}$$

If $|H| = \lfloor |V|/2 \rfloor$, then there are two cases:

- $|V|$ is even. Since H is a minimum maximal matching, every maximal matching of G is a perfect matching. Due to lemma 7, G is isomorphic to either K_{2n} or $K_{n,n} (n = |V|/2 > 2)$. For these graphs

$$c(G) = \frac{n-1}{n} > \frac{1}{n-1} = \frac{2}{|V|-2}$$

- $|V|$ is odd. If $|S_H| \geq 2$ then

$$c(G) \geq 2/|H| = 4/(|V|-1) \geq 2/(|V|-2) \tag{6}$$

  Assume $S_H = \{(u,v)\}$. Lemma 2 implies that either $|H| = 2(|V| = 5)$ or all the vertices of $V(H \backslash S_H)$ are connected to only one of the endpoints of (u,v). Without loss of generality we may assume that they are connected to u.

  If $|V| = 5$, there are only a few graphs for which it is possible to have $|H| = 2$ and $|S_H| = 1$. All these graphs can be easily checked to have a complexity of 1.
  Assume $|H| \geq 3$. Let w be the vertex, which is not covered by H. If w is connected to v then due to 1 of lemma 5 $|S_{H \cup \{(v,w)\} \backslash \{(u,v)\}}| > 1$, since all the edges of $H \cup \{(v,w)\} \backslash \{(u,v)\}$ are connected to u. As a result, according to (6),

$$c(G) > 2/(|V|-2).$$

  If w is connected to u, take an edge $(u_1, v_1) \in H$ such that $(u, u_1) \in E$. $H \cup \{(u, u_1)\} \backslash \{(u,v), (u_1, v_1)\})$ is a maximal matching with a smaller cardinality than H. Thus H is not minimum and this case is impossible.

The proof is now completed.

Figure 5 with $a = 0$ illustrates that the bound achieved in the previous theorem is tight. The depicted graph contains $2(b+2)$ vertices and has a complexity of $1/(b+1)$, therefore

$$c(G) = \frac{2}{|V(G)|-2}.$$

4.3. Bounds for the complexity of regular graphs

For regular graphs, it is possible to find lower bounds for their complexity that do not depend on the number of edges in those graphs.

Theorem 7 *For an r-regular graph G with $r > 1$ $c(G) \geq \frac{1}{2}$.*

Proof. Let H be a maximal matching. Let E_1 be the set of edges that connect $V(S_H)$ with $V(H \backslash S_H)$, E_2 be the set of edges that connect $V(S_H)$ with $V(G) \backslash V(H)$, and E_3 be the set of edges in the spanning subgraph of $V(S_H)$, not including the edges from S_H.

According to point 1 of lemma 5, all the vertices of $V(H\backslash S_H)$ are only connected to the vertices of $V(S_H)$. Therefore,

$$2|S_H|(r-1) = |V(S_H)|(r-1) = \sum_{v\in V(S_H)} (d(v)-1) = |E_1| + |E_2| + 2|E_3| \geq |E_1|$$
$$= \sum_{v\in V(H\backslash S_H)} (d(v)-1) = (r-1)|V(H\backslash S_H)| = 2|H\backslash S_H|(r-1).$$

Since $r \neq 1$, we have $|S_H| \geq |H\backslash S_H|$, thus $c(H) = |S_H|/|H| \geq \frac{1}{2} \Rightarrow c(G) \geq \frac{1}{2}$.

Corollary 4 *If G is a regular graph and $c(G) = \frac{1}{2}$ then for every maximal matching H, $c(H) = \frac{1}{2}$.*

Corollary 5 *If G is a regular graph and $c(G) = \frac{1}{2}$ then every maximal matching is a perfect matching.*

Proof. Since $c(H) = \frac{1}{2}$, we have that $|S_H| = |H\backslash S_H| \Rightarrow E_2 = \emptyset$. Now suppose there is vertex v, which is not covered by H. As H is maximal, it covers all the neighbors of v. Due to 1 of lemma 5, these neighbors cannot belong to $V(H\backslash S_H)$; consequently, they belong to $V(S_H)$. This contradicts with E_2 being empty.

Corollary 6 *The complexity of a regular graph G equals $\frac{1}{2}$ if and only if G is K_4 or $K_{2,2}$.*

Proof. It is not hard to see that $c(K_{2n}) = c(K_{n,n}) = \frac{n-1}{n}$. This said, the corollary follows from lemma 7 and corollary 5.

The following theorem shows that there exist better limits for the complexities of regular graphs if we do not consider graphs of small regularity.

Theorem 8 *For an r-regular graph G, we have*

(a) If $r > 4$ then $c(G) \geq \frac{2}{3}$;

(b) If $r = 4$ then $c(G) > \frac{3}{5}$.

Proof. (a) Due to lemma 2, for each $(u, v) \in S_H$ there are two options:

- u and v can be connected to the endpoints of only one edge from $H\backslash S_H$.
- u is not connected to any vertex covered by $H\backslash S_H$ and v may be connected to any number of endpoints of edges from $H\backslash S_H$.

Therefore, the edges of S_H are divided into two categories. Let A denote the set of edges of the first category, and B the set of the edges of the second category. If an edge from S_H falls in both categories, we will consider it to be in category A and not B.

Retaining the notations of the proof of theorem 7, we have $|E_1| = 2(r-1)|H\backslash S_H|$. The endpoints of each edge in category A are the endpoints of at most 4 edges from $|E_1|$. The

endpoints of each edge in category B are the endpoints of at most $r-1$ edges of E_1. This implies:

$$|E_1| \leq 4|A| + (r-1)|B| = (r-1)|S_H| - (r-5)|A| \leq (r-1)|S_H|.$$

We got that $2|H\backslash S_H| \leq |S_H|$, hence, $c(G) \geq c(H) = \frac{|S_H|}{|H|} \geq \frac{2}{3}$.

(b)We will assume that G is connected, because the case of disconnected graphs easily follows from the case of connected graphs. Choose any minimum maximal matching H of G.

Note that (2) of lemma 5 implies that if $e = (u, v) \in A$ then $u_1 = v_1$ or $(u_1, v_1) \in H\backslash S_H$. Moreover, $S_H = A \cup B, A \cap B = \emptyset$, and

$$|E_1| = 2(r-1)|H\backslash S_H| = 6|H\backslash S_H|.$$

The endpoints of each edge in category A are the endpoints of at most 4 edges from E_1, while the endpoints of each edge in category B are the endpoints of at most 3 edges of E_1. This implies:

$$6|H\backslash S_H| = |E_1| \leq 4|A| + 3|B| \leq 4|A| + 4|B| = 4|S_H|,$$

or

$$6|H| \leq 10|S_H|,$$

and therefore

$$c(H) = \frac{|S_H|}{|H|} \geq \frac{3}{5}. \tag{7}$$

Now, we claim that $c(H) > \frac{3}{5}$. If $c(H) = \frac{3}{5}$ then

$$|E_1| = 4|S_H| = 4|A|,$$

and therefore $B = \emptyset$. This implies that for each $e = (u, v) \in S_H$ there is exactly one $f = (u_1, v_1) \in H\backslash S_H$ such that

$$\{(u, u_1), (u, v_1), (v, u_1), (v, v_1), \} \subseteq E_1.$$

The uniqueness of f follows from lemma 6. Note that this correspondence is one-to-one since G is 4-regular and a $H\backslash S_H$ edge cannot be connected to two different A edges. Thus,

$$|H| = |S_H|,$$

and

$$c(H) = \frac{|S_H|}{|H|} = \frac{1}{2} < \frac{3}{5},$$

contradicting (7).

The proof is now completed.

Note that this bound is reachable, since K_6 is an 5-regular graph whose complexity is $\frac{2}{3}$.

Our interest toward the complexity and particularly, the complexity of regular graphs was motivated by the following

Conjecture 1 *If G is a connected regular graph with $c(G) < 1$, then G is either isomorphic to C_7 or there is $n, n \geq 2$ such that G is isomorphic either to $K_{n,n}$ or to K_{2n}, where C_7 is the cycle of length seven.*

In some sense, our conjecture states that all regular structures are "hard" except some "uninteresting" cases.

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5. Addendum

When the paper was completed, we realized that the statement of theorem 1 can be generalized to arbitrary clutters. The aim of this short section to prove the following

Theorem 9 *Let $L = (V, E)$ be an arbitrary clutter with $V = \{v_1, ..., v_n\}$, $E = \{U_1, ..., U_k\}$. Suppose that k does not divide n, and*

$$U_1 \cup ... \cup U_k = V, U_1 \cap ... \cap U_k = \emptyset.$$

Then

$$c(L) \geq \frac{1}{1 + n - 2\sqrt{n-1}}.$$

Proof. Again, we can assume that for each $U \in E$ $|S_U| = 1$. Consider the graph G_L, where $V(G_L) = V$, and

$$E(G_L) = \{(u, v) : \{u, v\} \text{ cannot be extended to } U \in E = E(L)\}.$$

Consider the vertices $v_1, ..., v_k$ with $S_{U_1} = \{v_1\}, ..., S_{U_k} = \{v_k\}$. Note that by definition of G_L, these vertices induce a maximum clique of G_L. Moreover, since $U_1 \cap ... \cap U_k = \emptyset$, we imply that each vertex lying outside Q, is adjacent to some vertex of Q. As, k does not divide n, there is $v_j \in V(Q)$ that is adjacent to at least $\lfloor \frac{n-k}{k} \rfloor + 1$ vertices lying outside Q. Clearly, for this vertex v_j we have:

$$d(v_j) \geq k - 1 + \lfloor \frac{n-k}{k} \rfloor + 1 = k + \lfloor \frac{n}{k} \rfloor - 1 \geq 2\sqrt{n-1} - 1.$$

Lemma 1 implies that there is $U \in U_{G_L}$ with

$$|U| \leq \frac{1}{1 + n - 2\sqrt{n-1}}.$$

It can be easily seen that since $v_j \in V(Q)$ then $U \in L$, thus

$$c(L) \geq c(U) \geq \frac{1}{1 + n - 2\sqrt{n-1}}.$$

Remark 6 *Note that in the proof of the theorem 9, our proof does not work if k divides n and for each $v \in V(Q)$ we have $d(v) = k - 1 + \frac{n-k}{k} = k + \frac{n}{k} - 2$. In that case, we have:*

$$c(L) = \frac{1}{2 + n - k - \frac{n}{k}}.$$

Remark 7 *As the following example shows, the condition that k does not divide n is important. Let $n = k^2$, $k \geq 2$. Consider a graph G obtained from a k-clique Q, and an independent set U_0 ($V(Q) \cap U_0 = \emptyset$) with $|U_0| = n - k = k^2 - k$, by joining every vertex of the clique to $k - 1$ vertices of U_0. Now, consider the clutter $L = U_G \backslash \{U_0\}$. Note that L satisfies all conditions of theorem 9 besides that k divides n. Now, it is not hard to see that*

$$c(L) = \frac{1}{n - k - \frac{n}{k} + 2} < \frac{1}{1 + n - 2\sqrt{n-1}}.$$